

Dirac Operator on the Quantum Sphere

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ABSTRACT

We construct a Dirac operator on the quantum sphere S_q^2 which is covariant under the action of $SU_q(2)$. It reduces to Watamuras' Dirac operator on the fuzzy sphere when $q \rightarrow 1$. We argue that our Dirac operator may be useful in constructing $SU_q(2)$ invariant field theories on S_q^2 following the Connes-Lott approach to noncommutative geometry.

Two important self-adjoint operators for the Connes-Lott approach to noncommutative geometry are the Dirac operator and chirality operator.[1],[2],[3] The former is needed to construct a differential calculus and the latter for chiral fermions. If symmetries are present on the noncommutative manifold, then these operators should respect the symmetries if they are to be applied in writing invariant quantum field theories. For example, the fuzzy sphere has an $SU(2)$ rotation symmetry which is reflected in the Dirac operator.[4],[5],[6],[7] Other noncommutative manifolds possessing different symmetry groups have been studied.[8] On the other hand, the program of Connes has not been extensively applied to noncommutative manifolds having symmetries associated with quantum groups. (In this regard, see [9].) In this letter we give a construction of the chirality operator Γ and Dirac operator D for a noncommutative manifold having an $SU_q(2)$ symmetry. The manifold is Podleś' quantum sphere S_q^2 . [10] It, along with the fuzzy sphere, has been shown to appear in certain sectors of string theory.[11] There are both finite and infinite dimensional representations for the algebra depending on the value of a certain parameter. Finite dimensional representations were given in [12]. We shall give an explicit construction of infinite dimensional representations here.

Besides the symmetry requirements on Γ and D , further conditions are: a) that Γ squares to unity, b) Γ commutes with the algebra \mathcal{A} associated with the noncommutative manifold, c) Γ and D anticommute and d) they yield the correct commutative limit. To understand what the correct commutative limit is in our case, we shall first review the spherically symmetric Dirac operator on S^2 . We then generalize to Dirac operators on a certain one parameter family of deformed (commutative) spheres. These deformed spheres are one point compactifications of certain Kähler manifolds. Their Dirac operators can also be regarded as rotationally invariant, but now with respect to the Poisson action of $SU(2)$. The Poisson action of $SU(2)$ on the deformed sphere is the commutative limit of the action of the quantum group $SU_q(2)$ on S_q^2 . So from our D defined on S_q^2 we should recover the Dirac operator on the deformed sphere in the commutative limit. Although the latter Dirac operator is rotationally invariant, the property of invariance is difficult to satisfy away from the commutative limit. Our D on S_q^2 is instead covariant. We argue that nevertheless it can be applied following Connes' scheme for writing $SU_q(2)$ invariant field theories on S_q^2 , which is of current interest[12]. Our construction of the Dirac operator and chirality operator is along the lines of the Watamuras' construction for the fuzzy sphere[5]*, and in fact, their Dirac operator and chirality operator are obtained from ours in a certain limit. † From our D one can thus

*Another Dirac operator on the fuzzy sphere was given by Grosse and Prešnajder[4].

†For a quite different construction, see [13]. Also, a Dirac operator on S_q^2 was given in [14] which did not have simple $SU_q(2)$ transformation properties. Its utility in writing invariant theories is therefore unclear.

construct differential calculi over a family of noncommutative spheres (parametrized by q), including the fuzzy sphere.

S². The Dirac operator \tilde{D} on S^2 can be expressed in different ways.[15] In terms of stereographically projected coordinates z and $\bar{z} = z^*$, it, along with the chirality operator $\tilde{\gamma}$, is given by

$$\tilde{D} = \eta^{-3/2} \begin{pmatrix} \partial & \\ -\bar{\partial} & \end{pmatrix} \eta^{1/2}, \quad \tilde{\gamma} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad (1)$$

where η is the conformal factor $\eta = (1 + |z|^2)^{-1}$, $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ and we assume unit radius. \tilde{D} and $\tilde{\gamma}$ are hermitean and anticommute with each other. Alternatively, the Dirac operator and chirality operator can be expressed in a rotationally invariant manner by going to three-dimensional embedded coordinates x_i , $x_i x_i = 1$, $i = 1, 2, 3$. For this one can apply a unitary transformation using

$$U = \eta^{1/2} \begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix}, \quad (2)$$

and $z = (x_1 - ix_2)/(2\eta)$, $\bar{z} = (x_1 + ix_2)/(2\eta)$. One gets

$$D = U \tilde{D} U^\dagger = \sigma_i \ell_i + 1, \quad (3)$$

$$\gamma = U \tilde{\gamma} U^\dagger = \sigma_i x_i, \quad (4)$$

where ℓ_i are the angular momenta $\ell_i = -i\epsilon_{ijk}x_j \frac{\partial}{\partial x_k}$ and σ_i are Pauli matrices. Yet another way to write D is in terms of the rotationally invariant Poisson structure on the sphere, i.e.

$$\{x_i, x_j\} = \epsilon_{ijk} x_k. \quad (5)$$

Acting on a spinor ψ it can be expressed by

$$-i D\psi = \{x \cdot \sigma, \psi\} - \frac{\{x \cdot \sigma, \{x \cdot \sigma, x \cdot \sigma\}\}}{2\{x \cdot \sigma, x \cdot \sigma\}} \psi \quad (6)$$

Deformed S². We now generalize the notion of rotational invariance to the include invariance with respect to Poisson actions of the rotation group. In this approach $SU(2)$ is a Poisson-Lie group. The most general Poisson structure on S^2 where the left action of $SU(2)$ on S^2 is a Poisson action is given by[16]

$$\{x_i, x_j\} = (1 - \lambda x_3) \epsilon_{ijk} x_k, \quad (7)$$

where λ is a real constant and we again assume $x_i x_i = 1$. Substituting into (6)[‡] gives the new Dirac operator

$$D = (1 - \lambda x_3)(\sigma_i \ell_i + 1) + \frac{i\lambda}{2} \epsilon_{ij3} x_i \sigma_j, \quad (8)$$

[‡]The inverse of $\{x \cdot \sigma, x \cdot \sigma\}$ is defined everywhere but $x_3 = 1/\lambda$.

which is a one parameter deformation of (3). Like (3), it is hermitean and anticommutes with the chirality operator (4). It too is invariant under simultaneous rotations of the spin and the coordinates, but now the latter is with a Poisson action. If we perform a unitary transformation using the inverse of (2) we recover (1) with the conformal factor η replaced by $\eta/(1 - \lambda x_3)$. From (7) this factor also appears in the symplectic two form, and as a result the manifold on which D is written is Kählerian, or more precisely a one point compactification of a Kähler manifold.

S_q^2 . For the generalization to quantum groups, we recall that Poisson actions are recovered from actions of quantum groups in the commutative limit. It is therefore natural to ask whether or not there is an extension of (8) to a Dirac operator which is invariant under the action of a quantum group. As stated earlier, the invariance requirement appears difficult to satisfy. On the other hand, we can find a Dirac operator with simple (covariant) transformation properties which is well suited for writing invariant field theories. The relevant quantum group is $SU_q(2)$, because the Poisson action of $SU(2)$ is recovered from the action of $SU_q(2)$ when $q \rightarrow 1$. $SU_q(2)$ has a natural action on the quantum sphere S_q^2 [10], which reduces to ordinary rotations on S^2 when $q \rightarrow 1$. We denote the generators of the algebra \mathcal{A} for S_q^2 by \mathbf{x}_+ , \mathbf{x}_- , \mathbf{x}_3 , along with the unit 1. They satisfy commutation relations

$$\begin{aligned} \mathbf{x}_+ \mathbf{x}_- - \mathbf{x}_- \mathbf{x}_+ &= \mu \mathbf{x}_3 - (q - q^{-1}) \mathbf{x}_3^2 \\ q \mathbf{x}_3 \mathbf{x}_+ - q^{-1} \mathbf{x}_+ \mathbf{x}_3 &= \mu \mathbf{x}_+ \\ q \mathbf{x}_- \mathbf{x}_3 - q^{-1} \mathbf{x}_3 \mathbf{x}_- &= \mu \mathbf{x}_- , \end{aligned} \quad (9)$$

subject to the constraint

$$\mathbf{x}_3^2 + q \mathbf{x}_- \mathbf{x}_+ + q^{-1} \mathbf{x}_+ \mathbf{x}_- = 1 \quad (10)$$

Not surprisingly, there are now two deformation parameters q and μ , which we take to be real (and we have again chosen the ‘radius’ equal to one). (9) and (10) are preserved under the involution $*$: $\mathbf{x}_\pm^* = \mathbf{x}_\mp$ and $\mathbf{x}_3^* = \mathbf{x}_3$.

There are two limits of interest of the relations (9) and (10). The deformed sphere is recovered when

$$q \rightarrow 1 \text{ and } \mu \rightarrow 0, \quad \text{with } \frac{q - q^{-1}}{\mu} \rightarrow \text{finite} , \quad (11)$$

which we refer to as the commutative limit. The limiting value of $(q - q^{-1})/\mu$ can be taken to be the constant λ appearing in (7) and (8). If we write $q = e^\tau$, then the commutation relations (9) at lowest order in τ become

$$[\mathbf{x}_i, \mathbf{x}_j] = \frac{-2i\tau}{\lambda} \{\mathbf{x}_i, \mathbf{x}_j\} + O(\tau^2), \quad i, j, \dots = 1, 2, 3 \quad (12)$$

where the Poisson brackets are those in (8), and

$$\mathbf{x}_1 = -\frac{1}{\sqrt{2}}(\mathbf{x}_+ + \mathbf{x}_-) \quad \mathbf{x}_2 = -\frac{i}{\sqrt{2}}(\mathbf{x}_+ - \mathbf{x}_-) . \quad (13)$$

Here $\frac{-2\tau}{\lambda}$ plays the role of \hbar . From (10) we get that $\mathbf{x}_i \mathbf{x}_i = 1 + O(\tau^2)$.

The other limit is

$$q \rightarrow 1 \quad \text{and} \quad \mu \rightarrow \pm \frac{1}{\sqrt{\ell(\ell+1)}} , \quad \ell = \frac{1}{2}, 1, \frac{3}{2}, \dots . \quad (14)$$

This is the limit of the fuzzy sphere associated with the $2\ell+1$ dimensional representation. Now (9) and (10) reduce to the familiar relations[2],[4],[5],[6],[7]

$$[\mathbf{x}_i, \mathbf{x}_j] = -\frac{i}{\sqrt{\ell(\ell+1)}} \epsilon_{ijk} \mathbf{x}_k , \quad \mathbf{x}_i \mathbf{x}_i = 1 , \quad (15)$$

where we again define \mathbf{x}_1 and \mathbf{x}_2 by (13). [§] If after taking the limit (14) we then take $\ell \rightarrow \infty$ we get back (undeformed) S^2 .

From the generators \mathbf{x}_i it is convenient to defined a 2×2 matrix $X = [X_{ab}]$ according to

$$X = \begin{pmatrix} q\mathbf{x}_3 & -\sqrt{\frac{[2]}{q}}\mathbf{x}_+ \\ -\sqrt{\frac{[2]}{q}}\mathbf{x}_- & -q^{-1}\mathbf{x}_3 \end{pmatrix} , \quad (16)$$

where

$$[n] = \frac{1 - q^{2n}}{1 - q^2} .$$

X has the properties:

- i) X is hermitean with respect to $*$, i.e. $X_{ab}^* = X_{ba}$,
- ii) It satisfies a deformed trace condition: $\text{Tr}_q X = q^{-1}X_{11} + qX_{22} = 0$,
- iii) $X^2 = \mathbb{1} + \mu X$, where $\mathbb{1}$ is the unit matrix.[¶]

The matrices X define an $SU_q(2)$ bimodule. $SU_q(2)$ matrices satisfy the commutation relations with themselves

$$\begin{matrix} R & T & T \\ 1 & 2 & 2 \end{matrix} = \begin{matrix} T & T & R \\ 2 & 1 & 1 \end{matrix} \quad (17)$$

[§]There exist deformations of these finite dimensional representations when $q \neq 1$. They occur for values of μ given in (43). [12],[10]

[¶]Using this property one can construct projection operators $P_{\pm} = \frac{1}{2}\{\mathbb{1} \pm (X - \frac{\mu}{2})/\sqrt{1 + \frac{\mu^2}{4}}\}$. They are the magnetic monopoles projectors of [17]. (We thank Tomasz Brzezinski for this remark.) They reduce to the magnetic monopoles projectors for the fuzzy sphere[6] in the limit (14).

where $T_1 = T \otimes \mathbb{1}$, $T_2 = \mathbb{1} \otimes T$ and

$$R = \begin{pmatrix} q & & & \\ & 1 & & \\ & q - q^{-1} & 1 & \\ & & & q \end{pmatrix}, \quad (18)$$

and R fulfills the quantum Yang-Baxter equation. In addition, T satisfies a unitarity condition (using the involution $*$) and also a deformed unimodularity condition $\det_q T = 1$, where $\det_q T = T_{11}T_{22} - qT_{12}T_{21}$. This constraint is possible because $\det_q T$ so defined is in the center of the algebra. Under the action of $SU_q(2)$ X undergoes a similarity transformation

$$X \rightarrow X' = TXT^{-1}, \quad (19)$$

which preserves the relations i)-iii). Thus (19) is an algebra homomorphism. Although matrix elements of T do not commute amongst themselves, they are assumed to commute with \mathcal{A} . There is an analogue of the cyclic property of the trace (now with respect to Tr_q) for the matrices T and this leads to ii) being preserved under $SU_q(2)$.

In either the commutative limit or the fuzzy limit, X reduces to $\sigma_i x_i$, and the analogue of transformation (19) rotates either the coordinates or the spin matrices:

$$\sigma_i x_i \rightarrow g \sigma_i g^\dagger x_i = \sigma_j \theta_{ji} x_i, \quad (20)$$

where $g \in SU(2)$ and θ is the corresponding rotation matrix. Alternatively, we can write

$$\sigma_i x_i = g \sigma_i g^\dagger \theta_{ij}^{-1} x_j, \quad (21)$$

and say that $\sigma_i x_i$ is invariant with respect to simultaneous rotations of the coordinates and the spin generated by the total angular momentum. There is no analogue of this statement with regard to $SU_q(2)$ transformations of X . For this define the deformed Pauli matrices:

$$\sigma_3^q = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix} \quad \sigma_+^q = \begin{pmatrix} & -\sqrt{\frac{[2]}{q}} \\ & \end{pmatrix} \quad \sigma_-^q = \begin{pmatrix} & \\ -\sqrt{\frac{[2]}{q}} & \end{pmatrix}, \quad (22)$$

and write $X = \sigma_3^q \mathbf{x}_3 + \sigma_+^q \mathbf{x}_+ + \sigma_-^q \mathbf{x}_- = \sigma_i^q \mathbf{x}_i$. Then (19) becomes

$$\sigma_i^q \mathbf{x}_i \rightarrow T \sigma_i^q T^{-1} \mathbf{x}_i = \sigma_j^q \Theta_{ji} \mathbf{x}_i, \quad (23)$$

Θ_{ij} giving the spin one representation of $SU_q(2)$ and $\mathbf{x} \rightarrow \Theta_{ij} \mathbf{x}_j$ defines the algebra homomorphism for S_q^2 . The analogue of (21) is true, namely,

$$\sigma_i^q \mathbf{x}_i = T \sigma_i^q T^{-1} \Theta_{ij}^{-1} \mathbf{x}_j, \quad (24)$$

but Θ_{ij}^{-1} is not in the spin one representation of $SU_q(2)$, and $\mathbf{x} \rightarrow \Theta_{ij}^{-1} \mathbf{x}_j$ is not an algebra homomorphism for S_q^2 . For this reason we cannot argue that X is invariant under simultaneous $SU_q(2)$ transformations of the coordinates and the spin. We can only regard it as covariant with respect to transformations of either the noncommuting coordinates or the deformed spin matrices. Since we will construct the Dirac operator and chirality operator for S_q^2 from X , the same will apply for these operators.

In analogy with the Watamuras' construction for the fuzzy sphere, [5] we introduce an \mathcal{A} -bimodule \mathcal{M} , whose elements belong to \mathcal{A} . It is acted on from the left by \mathcal{A}_L and the right by \mathcal{A}_R . Elements a_L, b_L, \dots of \mathcal{A}_L satisfy the same algebra as \mathcal{A} , i.e. $a_L b_L = (ab)_L$, $a, b, \dots \in \mathcal{A}$, while elements a_R, b_R, \dots of \mathcal{A}_R satisfy $a_R b_R = (ba)_R$. The action of an element $a_L \otimes b_R$ of $\mathcal{A}_L \otimes \mathcal{A}_R$ on \mathcal{M} is given by

$$(a_L \otimes b_R) \circ \mathcal{M} = a \mathcal{M} b \quad (25)$$

It follows that left and right variables commute, i.e. $[a_L \otimes 1, 1 \otimes b_R] = 0$. Then we can fulfill the requirement that Γ commutes with $\mathcal{A} = \mathcal{A}_L$ by constructing it from only from elements of \mathcal{A}_R .

Next we introduce a spin structure. We define spinor fields Ψ to take values in $\mathcal{A} \otimes C^2$, where C^2 is the space of two dimensional spinors, and to transform covariantly under $SU_q(2)$, i.e.

$$\Psi \rightarrow \Psi' = T \Psi, \quad (26)$$

They are acted on by operators \mathcal{O} belonging to $\mathcal{A}_L \otimes \mathcal{A}_R \otimes M^2$, where M^2 is the space of 2×2 matrices. For the result to be a spinor we need that

$$(\mathcal{O} \circ \Psi) \rightarrow \mathcal{O}' \circ \Psi' = T(\mathcal{O} \circ \Psi) \quad (27)$$

under $SU_q(2)$. It is easy to find a solution to (27) for arbitrary spinors for the case where \mathcal{O} has only trivial dependence in \mathcal{A}_R . Then one can just write $\mathcal{O} = \mathbf{L}$, where

$$\mathbf{L} \circ \Psi = X \Psi \quad (28)$$

and use (19). For the case where \mathcal{O} has only trivial dependence in \mathcal{A}_L , one can define $\mathcal{O} = \mathbf{R}$, where

$$(\mathbf{R} \circ \Psi)^T = \Psi^T \epsilon X \epsilon, \quad (29)$$

The superscript T denotes transpose and

$$\epsilon = \begin{pmatrix} & 1 \\ -q & \end{pmatrix}. \quad (30)$$

Using (19) and the identities

$$T \epsilon T^T = T^T \epsilon T = \epsilon \quad (31)$$

one can show that X_R satisfies the covariance condition (27):

$$\begin{aligned}
(\mathbf{R}' \circ \Psi')^T &= \Psi'^T \epsilon X' \epsilon \\
&= \Psi^T T^T \epsilon T X T^{-1} \epsilon \\
&= \Psi^T \epsilon X \epsilon T^T \\
&= (T (\mathbf{R} \circ \Psi))^T
\end{aligned} \tag{32}$$

From (28) and (29), \mathbf{L} and \mathbf{R} have matrix elements

$$\mathbf{L}_{ab} = (X_{ab})_L \quad \mathbf{R}_{ab} = \epsilon_{bc} \epsilon_{da} (X_{cd})_R \tag{33}$$

Finally, one can construct $SU_q(2)$ covariant operators \mathcal{O} with a nontrivial dependence in both \mathcal{A}_L and \mathcal{A}_R by taking matrix products of \mathbf{L} and \mathbf{R} .

We can now define the chirality operator Γ . Γ is defined to square to the identity and commute with \mathcal{M} . As remarked earlier, Γ should be trivial in \mathcal{A}_L . In addition, we require Γ to satisfy the covariance condition (27) with respect to $SU_q(2)$ transformations. The solution for Γ is then

$$\Gamma = \frac{1}{q\sqrt{4+\mu^2}} (2\mathbf{R} + q\mu \mathbb{1}) . \tag{34}$$

Its matrix elements are

$$\Gamma_{ab} = \frac{1}{q\sqrt{4+\mu^2}} (2\epsilon_{bc} \epsilon_{da} (X_{cd})_R + q\mu \delta_{ab}) . \tag{35}$$

The Dirac operator D is required to anticommute with Γ . We shall also demand that it transform covariantly under $SU_q(2)$. These requirements are met for any D of the form

$$D = \Gamma [Y, \Gamma] , \tag{36}$$

where Y transform covariantly under $SU_q(2)$. For D to have a nontrivial commutator with \mathcal{M} , Y should be nontrivial in \mathcal{A}_L . A natural choice is therefore

$$Y = \frac{1}{2\mu} \mathbf{L} , \tag{37}$$

The factor $\frac{1}{2\mu}$ was inserted to get a finite commutative limit, defined in (11), which we compute below. We show that if we once again choose the limiting value of $(q - q^{-1})/\mu$ to be the constant λ appearing in (7) and (8) we recover the Dirac operator (8) for the deformed sphere. If on the other hand we take the fuzzy sphere limit (14), D reduces to the Dirac operator in [5]. For arbitrary values of q and μ , the matrix elements of D are

$$D_{ab} = \frac{1}{2q^2\mu(4+\mu^2)} (X_{cd})_L \otimes \left((2\epsilon X \epsilon + q\mu \mathbb{1})_{bd} (2\epsilon X \epsilon + q\mu \mathbb{1})_{ca} - q^2(4+\mu^2) \delta_{bd} \delta_{ca} \right)_R . \tag{38}$$

We now show that the commutative limit of Γ and D is (4) and (8), respectively. As earlier, we set $q = e^\tau$ and $\mu = 2\tau/\lambda + O(\tau^2)$ and then perform an expansion in τ . Up to first order

$$X = \mathbf{x}_i \sigma_i + \tau \mathbf{x}_3 \mathbb{1} + O(\tau^2)$$

$$\epsilon X \epsilon = \mathbf{x}_i \sigma_i^T + \tau \left(\mathbf{x}_3 (\sigma_3 - \mathbb{1}) + (\mathbf{x}_1 - i \mathbf{x}_2) (\sigma_1 - i \sigma_2) \right) + O(\tau^2), \quad (39)$$

where σ_i , $i = 1, 2, 3$, are Pauli matrices and \mathbf{x}_1 and \mathbf{x}_2 were defined in (13). Then

$$\frac{1}{q\sqrt{4+\mu^2}} (2\epsilon X \epsilon + q\mu \mathbb{1}) = \mathbf{x} \cdot \sigma^T + \tau \left(i\epsilon_{ij3} \mathbf{x}_i \sigma_j^T + (\lambda^{-1} - \mathbf{x}_3) \mathbb{1} \right) + O(\tau^2). \quad (40)$$

We only need the zeroth order term to show that the commutative limit of Γ is (4), while we need the first order term to compute D . Up to first order in τ , D acting on a spinor ψ is given by

$$\begin{aligned} 2\mu D_{ab} \circ \psi_b &\rightarrow (\mathbf{x} \cdot \sigma)_{cd} \psi_b (\mathbf{x} \cdot \sigma)_{db} (\mathbf{x} \cdot \sigma)_{ac} - (\mathbf{x} \cdot \sigma)_{ab} \psi_b \\ &\quad + 2\tau \left(i\epsilon_{ij3} \mathbf{x}_i \sigma_j \psi + (\lambda^{-1} - \mathbf{x}_3) \psi \right)_a, \end{aligned} \quad (41)$$

as $q \rightarrow 1$. We have neglected the ordering of factors of \mathbf{x}_i and ψ_a in the first order terms, which is not valid at zeroth order. For the latter, we can use (12) with Poisson brackets given in (8). This gives

$$2\mu D_{ab} \circ \psi_b \rightarrow 4\tau(\lambda^{-1} - \mathbf{x}_3)(-i\epsilon_{ijk} \sigma_i \mathbf{x}_j \partial_k \psi + \psi)_a + 2\tau i\epsilon_{ij3} \mathbf{x}_i (\sigma_j \psi)_a, \quad (42)$$

and consequently (8).

In order to proceed with Connes' construction of the differential calculus, one must obtain the spectra of the Dirac operator and introduce a Hilbert space for the spinors Ψ . We have not yet attempted the former. Concerning the latter, there are both finite and infinite dimensional Hilbert spaces. The finite dimensional Hilbert spaces occur for certain discrete values of μ :

$$\mu = \pm \frac{[2(2\ell+1)]}{q[2\ell+1]} \frac{1}{\sqrt{[2\ell][2\ell+2]}}, \quad \ell = \frac{1}{2}, 1, \frac{3}{2}, \dots, \quad (43)$$

where the dimension is $2(2\ell+1)$. (The factor 2 is because we have spinors.) They were explicitly constructed in [12]. In the limit $q \rightarrow 1$, (43) goes to (14) and the corresponding matrix representations for the algebra agree with those of the fuzzy sphere.

Infinite dimensional representations occur for

$$\mu = q - q^{-1}, \quad (44)$$

which can emerge as the large ℓ limit of (43). A possible construction involves the use of coherent states. Here one can adopt the approach developed in [18],[19] which relies on coherent states for deformed creation and annihilation operators, $\tilde{\mathbf{a}}^\dagger$ and $\tilde{\mathbf{a}}$, respectively. They satisfy the commutation relations

$$[\tilde{\mathbf{a}}, \tilde{\mathbf{a}}^\dagger] = F(\tilde{\mathbf{a}}\tilde{\mathbf{a}}^\dagger) \quad (45)$$

for some function F . The deformed coherent states diagonalize $\tilde{\mathbf{a}}$, and have a natural scalar product. An explicit construction was given in [19] for the fuzzy sphere. There one identified $\tilde{\mathbf{a}}^\dagger$ and $\tilde{\mathbf{a}}$ with the fuzzy analogues of stereographic coordinates. Something similar can be done for the quantum sphere, since analogues of stereographic coordinates also exist for S_q^2 . [14] This construction, however, only works when (44) is satisfied. Then one can parametrize the matrix X according to

$$X = \begin{pmatrix} q(1 - [2]\eta) & q^{-1}[2]\mathbf{z}\eta \\ q^{-1}[2]\eta\bar{\mathbf{z}} & q^{-1}([2]\eta - 1) \end{pmatrix}, \quad (46)$$

where $\bar{\mathbf{z}} = \mathbf{z}^*$, and we assume that the operator $1 + \bar{\mathbf{z}}\mathbf{z}$ is nonsingular, with

$$\eta^{-1} = 1 + \bar{\mathbf{z}}\mathbf{z} = q^2(1 + \mathbf{z}\bar{\mathbf{z}}), \quad (47)$$

which gives the commutation relations for \mathbf{z} and $\bar{\mathbf{z}}$. From these relations one can verify i-iii). Now identify: $\tilde{\mathbf{a}} = \mathbf{z}$, $\tilde{\mathbf{a}}^\dagger = \bar{\mathbf{z}}$ and $F(\mathbf{z}\bar{\mathbf{z}}) = -q^{-1}\mu\eta^{-1}$, where we used (44). To write the deformed coherent states one introduces a map from a pair of standard (or undeformed) creation and annihilation operators, \mathbf{a}^\dagger and \mathbf{a} , satisfying

$$[\mathbf{a}, \mathbf{a}^\dagger] = 1, \quad (48)$$

to $\tilde{\mathbf{a}}^\dagger$ and $\tilde{\mathbf{a}}$. The map can be expressed as

$$\tilde{\mathbf{a}} = f(\mathbf{n} + 1) \mathbf{a}, \quad (49)$$

\mathbf{n} being the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$, having eigenvalues $n = 0, 1, 2, \dots$. The function f is determined from F . We get

$$|f(n)|^2 = -q^{-2n+1}\mu\frac{[n]}{n}, n > 0. \quad (50)$$

For $n = 0$ we can take the limiting value $|f(0)|^2 = -2\ln q$. For the right hand side of (50) to be positive we must restrict $q < 1$. We note that the undeformed creation and annihilation operators are not recovered in the commutative limit, i.e. $q \rightarrow 1$, and further f is ill-defined in the limit. The Hilbert space \mathbb{H} can now be defined as being spanned by the eigenstates $|n\rangle$, $n = 0, 1, 2, \dots$, of \mathbf{n} with scalar product $\langle n|m \rangle = \delta_{n,m}$. (Actually,

we want two copies of \mathbb{H} for the spinors.) Alternatively, one can use the overcomplete coherent state basis: [18],[19]

$$\begin{aligned} |\zeta\rangle &= N(|\zeta|^2)^{-\frac{1}{2}} \exp\{\zeta f(\mathbf{n})^{-1} \mathbf{a}^\dagger\} f(\mathbf{n})^{-1} |0\rangle \\ &= N(|\zeta|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{n!} [f(n)]!} |n\rangle, \end{aligned} \quad (51)$$

where $[f(n)]! = f(n)f(n-1)\dots f(0)$, which diagonalize $\tilde{\mathbf{a}}$. Requiring $|\zeta\rangle$ to be of unit norm fixes $N(|\zeta|^2)$,

$$N(x) = \sum_{n=0}^{\infty} \frac{x^n}{n! ([f(n)]!)^2}. \quad (52)$$

As with the standard coherent states, the states (51) are not orthonormal, but instead satisfy

$$\langle \eta | \zeta \rangle = N(|\eta|^2)^{-\frac{1}{2}} N(|\zeta|^2)^{-\frac{1}{2}} N(\bar{\eta}\zeta) \quad (53)$$

(Alternatively, a construction of finite dimensional Hilbert spaces exists for certain discrete values of μ [12].)

From the above, we conclude that when (44), along with $q < 1$, are satisfied there is an infinite dimensional Hilbert space which should allow for a construction of a differential calculus on S_q^2 following Connes. We recall that the condition (44) was previously found to be necessary for writing down a differential calculus on S_q^2 in a very different approach. [20] The differential calculus of [20] was obtained by demanding invariance of the exterior derivative d under $SU_q(2)$. The corresponding algebra of one forms is easily expressed in terms of \mathbf{z} and $\bar{\mathbf{z}}$ and their exterior derivatives:[14]

$$\begin{aligned} \mathbf{z}d\mathbf{z} &= q^{-2}d\mathbf{z}\mathbf{z}, & \mathbf{z}d\bar{\mathbf{z}} &= q^{-2}d\bar{\mathbf{z}}\mathbf{z} \\ \bar{\mathbf{z}}d\mathbf{z} &= q^2d\mathbf{z}\bar{\mathbf{z}}, & \bar{\mathbf{z}}d\bar{\mathbf{z}} &= q^2d\bar{\mathbf{z}}\bar{\mathbf{z}} \\ d\mathbf{z}d\bar{\mathbf{z}} &= -q^{-2}d\bar{\mathbf{z}}d\mathbf{z}, & (d\mathbf{z})^2 &= (d\bar{\mathbf{z}})^2 = 0 \end{aligned} \quad (54)$$

We, on the other hand, do not recover these formulae upon representing $d\mathbf{z}$ and $d\bar{\mathbf{z}}$ with $[D, \mathbf{z}]$ and $[D, \bar{\mathbf{z}}]$, respectively, following Connes. This cannot be surprising since our Dirac operator is only covariant and therefore, in contrast to [20], will not lead to an invariant exterior derivative. (From the Dirac operator of [14] one does recover the Podles differential calculus (54), however, that Dirac operator does not have simple transformation properties under $SU_q(2)$.)

Although our exterior derivative is not invariant, it should nevertheless be useful for writing invariant field theories[12]. For example, if $\phi \in \mathcal{A}$ represents a scalar field on S_q^2 , we can construct the following quadratic invariant

$$\text{Tr}_q[D, \phi][D, \phi], \quad (55)$$

since $\text{Tr}_q TMT^{-1} = \text{Tr}_q M$ for any $M \in \mathcal{A} \times M^2$. For fields belonging to nontrivial representations of $SU_q(2)$ additional deformed traces can be taken. In order to be useful in constructing actions, we will also have to take a trace in the Hilbert space, which will require clarification. More exciting is the possibility that q can be made into a dynamical quantity, thereby introducing dynamics in the underlying noncommuting manifold, and possibly allowing for quantum fluctuations about fuzzy spheres.

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REFERENCES

- [1] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [2] J. Madore, *An Introduction to Noncommutative Differential Geometry and its Applications*, Cambridge University Press, Cambridge, 1995.
- [3] G. Landi, *An Introduction to Noncommutative Spaces And Their Geometries*, Springer-Verlag, Berlin, 1997.
- [4] H. Grosse and P. Prešnajder, Lett.Math.Phys. **33**, 171 (1995).
- [5] U. Carow-Watamura, S. Watamura, Commun. Math. Phys. **183**, 365 (1997); Int. J. Mod. Phys. **A13**, 3235 (1998); Commun. Math. Phys. **212**, 395 (2000).
- [6] S. Baez, A.P. Balachandran, B. Idri, S. Vaidya, Commun.Math.Phys. **208**, 787 (2000); A.P. Balachandran, T.R. Govindarajan, B. Ydri, hep-th/9911087; Mod.Phys.Lett. **A15**, 1279 (2000); A.P. Balachandran, S. Vaidya, hep-th/9910129; A.P. Balachandran, X. Martin, D.O'Connor, hep-th/0007030.
- [7] H. Grosse, C. Klimcik, P. Prešnajder, Commun.Math.Phys.**178**, 507 (1996); **180**, 429 (1996); H. Grosse and P. Prešnajder, Lett.Math.Phys. **46**, 61 (1998).
- [8] H. Grosse and A. Strohmaier, Lett.Math.Phys. **48** 163 (1999); G. Alexanian, A.P. Balachandran, G. Immirzi, B. Ydri, hep-th/0103023.
- [9] M. Paschke, A. Sitarz, Acta Phys.Polon. **B31**, 1897 (2000).
- [10] P. Podles, Lett. Math. Phys. **14**, 193 (1987).
- [11] A. Yu. Alekseev, A. Recknagel, V. Schomerus, JHEP 9909:023,1999.
- [12] H. Grosse, J. Madore, H. Steinacker, hep-th/0005273; hep-th/0103164.

- [13] K. Ohta and H. Suzuki , hep-th/9405180.
- [14] C.-S. Chu, P.-M. Ho and B. Zumino, Cargese 1996, “Quantum fields and quantum space time”, 281-322, hep-th/9608188 .
- [15] C. Jayewardena, Helv. Phys. Acta. **61**, 636 (1988).
- [16] A. J.-L. Sheu, J.-H. Lu and A. Weinstein, Comm. Math. Phys. **135** , 217 (1991).
- [17] T. Brzezinski and S. Majid, Comm. Math. Phys. **213** (2000) 491; math.qA/9807052.
- [18] V.I. Man’ko, G. Marmo, E.C.G. Sudarshan, F. Zaccaria, Physica Scripta **55**, 528 (1997).
- [19] G. Alexanian, A. Pinzul and A. Stern, Nucl. Phys. **B600**, 531 (2001), hep-th/0010187.
- [20] P. Podles, Lett. Math. Phys. **18**, 107 (1989) ; Comm. Math. Phys. **150**, 167 (1992).